ON THE EQUATIONS OF THE UNPERTURBED WORK OF AN INERTIAL SYSTEM DETERMINING CURVILINEAR COORDINATES

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An inertial system independently defining the coordinates and orientation of an object in space by means of accelerometers and gyroscopes [1 and 2] is investigated. The general case of determination of arbitrary, nonstationary and nonorthogonal curvilinear coordinates is considered, as distinct from [2] in which these coordinates were Cartesian. Equations are devided for the unperturbed functioning of such an inertial system with its kinematic model based on a gyro-stabilized platform, or a controllable gyro-frame.

1. We introduce a right-hand orthogonal system of coordinates $O_1\xi^1\xi^2\xi^3$ with its origin at the center of the Earth, and its axes permanently oriented in the direction of fixed stars. The position of the moving object will be determined by the position of any of its point O in relation to $O_1\xi^1\xi^2\xi^3$ by coordinates $\varkappa^1, \varkappa^2, \varkappa^3$, so that

$$\xi^s = \xi^s(\varkappa^1, \varkappa^2, \varkappa^3, t), \qquad J = \frac{D(\xi^1, \xi^2, \xi^3)}{D(\varkappa^1, \varkappa^2, \varkappa^3)} \neq 0 \tag{1.1}$$

Here and in the following text, Latin indexes run from 1 to 3. According to the second expression of (1.1) the Jacobian of transformation of function ξ^s with respect to coordinates $\varkappa^n$ is different from zero. This means that relationships (1.1) are reversible throughout the region of possible motions of the object.

The model of this inertial guidance system is visualized as follows. At its base is a gyro-stabilized platform, the axes of which coincide with the directions of axes ξ^s of the coordinate system $O_1\xi^1\xi^2\xi^3$. The platform carries three accelerometers set to a special pattern. The unit vectors of direction of the sensitivity axes of the accelerometers will be denoted by $\mathbf{e}_s$, and their indications by n_{e_s}. We shall assume the kinematic pattern to be such, as to provide the required dependence of directions $\mathbf{e}_s$ from coordinates $\varkappa^s$, determined by the inertial system and from time t :

$$\mathbf{e}_s = e_s(\varkappa^1, \varkappa^2, \varkappa^3, t).$$

We assume that directions $\mathbf{e}_s$ are not coplanar.

We shall introduce a fundamental coordinate base [3 and 4] defined by vectors

$$\mathbf{r}_s = \frac{\partial \mathbf{r}}{\partial \varkappa^s}, \qquad \mathbf{r} = \xi^s \boldsymbol{\xi}_s \tag{1.2}$$

where $\mathbf{r}$ is the radius vector from the center of the Earth O_1 to point O of the object, and $\boldsymbol{\xi}_s$ are unit vectors of axes ξ^s. The Latin superscripts and subscripts in the second equation of (1.2) indicate here and in the following text summation in s from 1 to 3. It follows from (1.1') that vectors $\mathbf{r}_s$ are not coplanar. We shall introduce a reciprocal base defined by vectors $\mathbf{r}^s$ reciprocal to vectors $\mathbf{r}_s$.

Finally, we shall introduce the metric tensor A of the space defined by the curviliner coordinates $\varkappa^s$, and shall denote by a_{sk}, a^{sk}, a_s^k, the covariant, contravariant and mixed components of the tensor, so that

$$a_{sk} = \mathbf{r}_s \cdot \mathbf{r}_k, \qquad a^{sk} = \mathbf{r}^s \cdot \mathbf{r}^k, \qquad a_s^k = \mathbf{r}_s \cdot \mathbf{r}^k \tag{1.3}$$

The indications of accelerometers positioned along directions $\mathbf{e}_s$ will be [2]

$$n_{e_s} = \mathbf{n} \cdot \mathbf{e}_s, \qquad \mathbf{n} = \frac{d^2\mathbf{r}}{dt^2} - \mathbf{g}(\mathbf{r}) \tag{1.4}$$

Here $\mathbf{g}(\mathbf{r})$ is the intensity of the Earth's gravitational field at point O which characterizes the present position of the object and is determined by the radius vector $\mathbf{r}$.

Let $\mathbf{v}$ and $\mathbf{w}$ denote the absolute velocity and acceleration of point O with respect to the $O_1\xi^1\xi^2\xi^3$ system of coordinates. The covariant and contravariant components of vector $\mathbf{v}$ are

$$v^s = \varkappa^{s\cdot} + \frac{\partial \mathbf{r}}{\partial t} \cdot \mathbf{r}^s, \qquad v_s = a_{sk} v^k \tag{1.5}$$

where the dot indicates differentiation with respect to time.

Differentiation of vector $\mathbf{v} = \mathbf{r}_s v^s$ with respect to time gives

$$\mathbf{w} = \mathbf{r}_s \varkappa^{s\cdot\cdot} + \frac{\partial \mathbf{r}_s}{\partial \varkappa^k} \varkappa^{s\cdot} \varkappa^{k\cdot} + 2\frac{\partial \mathbf{r}_s}{\partial t} \varkappa^{s\cdot} + \frac{\partial^2 \mathbf{r}}{\partial t^2} \tag{1.6}$$

In order to establish expressions for components w_s and w^s of vector $\mathbf{w}$ we shall use Christoffel symbols $\Gamma_{sk,m}$ of the first, and Γ_{sk}^m of the second kind [3 and 4] and also symbols $\Gamma_{00,s}$, $\Gamma_{0k,s}$ and Γ_{00}^s, Γ_{0k}^s, defined as follows:

$$\Gamma_{00,s} = \frac{\partial^2 \mathbf{r}}{\partial t^2} \cdot \mathbf{r}_s, \quad \Gamma_{0k,s} = \frac{\partial^2 \mathbf{r}}{\partial t \partial \varkappa^k} \cdot \mathbf{r}_s, \quad \Gamma_{00}^s = a^{sn}\Gamma_{00,n}, \quad \Gamma_{0k}^s = a^{sn}\Gamma_{0k,n} \tag{1.7}$$

Although these symbols are in their meaning analogous to Christoffel's, they differ from the latter in that they are not expressed in terms of components of the metric tensor. Symbols $\Gamma_{00,s}$, $\Gamma_{0k,s}$ and Γ_{00}^s, Γ_{0k}^s characterize the nonstationary state of the $\varkappa^s$ reference grid. These symbols become zero if the right-hand sides of the first set of Equations (1.1) do not explicitly depend on time.

From (1.6), (1.7) and the definition of Christoffel symbols we have

$$w^s = \varkappa^{s\cdot\cdot} + \Gamma_{mn}^{s}\varkappa^{m\cdot}\varkappa^{n\cdot} + 2\Gamma_{0n}^{s}\varkappa^{n\cdot} + \Gamma_{00}^{s}, \qquad w_s = a_{sr}w^r \tag{1.8}$$

For stationary coordinates, Formulas (1.8) can be naturally converted to the usual formulas for covariant differentiation of vector $\mathbf{v}$.

Now, from (1.4) and (1.8)

$$n^s = \varkappa^{s\cdot\cdot} + \Gamma_{mn}^{s}\varkappa^{m\cdot}\varkappa^{n\cdot} + 2\Gamma_{0n}^{s}\varkappa^{n\cdot} + \Gamma_{00}^{s} - g^s, \qquad n_s = a_{sr}n^r \tag{1.9}$$

Here n^s, g^s, n_s, g_s are the contravariant and covariant components of vectors $\mathbf{n}$ and $\mathbf{g}$ in the basic system.

Vector $\mathbf{g}$ is given in the coordinate system tied to the Earth. It can be considered as given in the $O_1\xi^1\xi^2\xi^3$ system only on the assumption of sphericity of the Earth's gravitational field.

We shall introduce a system of coordinates $O_1\eta^1\eta^2\eta^3$ rigidly tied to the Earth. Its relation to the $O_1\xi^1\xi^2\xi^3$ system will be defined by direction cosines $\alpha_{ij}\ (= \alpha^{ij} = \alpha_i^j)$, so that the unit vectors η_s of η^s-axes are

$$\eta_s = \alpha_s^n \xi_n \tag{1.10}$$

In the system of coordinates $O_1, \eta^1\eta^2\eta^3$

$$\mathbf{g} = \mathrm{grad}\ U, \qquad U = U(\eta^1, \eta^2, \eta^3) \tag{1.11}$$

where U is the function of force of the gravitational field. Therefore

$$g_s = \mathrm{grad}^l\ U\eta_{ls}, \qquad g^s = \mathrm{grad}^l\ U\eta_l^s \tag{1.12}$$

In these equations η_{ls} and η_l^s denote the covariant and contravariant components of locus of $\boldsymbol{\eta}_l$ in the basic system.

From Formulas (1.4),(1.9) and (1.12) we find the following expressions for values indicated by the accelerometers

$$n_{e_s} = (\varkappa^{k\cdot\cdot} + \Gamma_{mn}^{k}\varkappa^{m\cdot}\varkappa^{n\cdot} + 2\Gamma_{0n}^{k}\varkappa^{n\cdot} + \Gamma_{00}^{k} - \mathrm{grad}^l U\eta_l^k)\, e_{sk} \tag{1.13}$$

Here e_{sk} denotes the covariant components of vector $\mathbf{e}_s$ with respect to vectors $\mathbf{r}_s$.

2. We shall now derive equations of work of the inertial system. The problem is to determine coordinates $\varkappa^s$ from indications of accelerometers as given by (1.13).

The first operation is to integrate accelerometer indications [2]. From (1.13) we obtain

$$\varkappa^{k\cdot}e_{sk} = \int_0^t [n_{e_s} + \varkappa^{k\cdot}\dot{e}_{sk} - (\Gamma_{mn}^{k}\varkappa^{m\cdot}\varkappa^{n\cdot} + 2\Gamma_{0n}^{k}\varkappa^{n\cdot} + \Gamma_{00}^{k} - \mathrm{grad}^l U\eta_l^k) e_{sk}]\ dt + $$
$$+ \varkappa^{k\cdot}(0)\, e_{sk}(0) \tag{2.1}$$

Resolving the left-hand side of Equation (2.1) with respect to $\varkappa^{k\cdot}$, we get

$$\varkappa^{k\cdot} = (\varkappa^{m\cdot}e_{sm})\, E^{sk} / E \tag{2.2}$$

Here E is the determinant, and E^{sk} the algebraic complement of row s and column k of matrix $\|e_{sk}\|$. Now

$$\varkappa^k = \int_0^t \varkappa_{;}^{k\cdot} dt + \varkappa^k(0) \qquad (2.3)$$

If instead of e_{sk} we have, as functions of $\varkappa^s$ and time the direction cosines γ_s^l, of vectors $\mathbf{e}_s$ with respect to the axes of the gyro-stabilized platform, i.e. in relation to loci $\boldsymbol{\xi}_s$ of the system of coordinates $O_1\xi^1\xi^2\xi^3$, we obtain $\mathbf{e}_s = \gamma_s^l \boldsymbol{\xi}_l$.

On the other hand $\mathbf{r}_k = (\partial\xi^l / \partial\varkappa^k)\,\boldsymbol{\xi}_l$. Therefore

$$e_{sk} = \mathbf{e}_s \cdot \mathbf{r}_k = \gamma_s^l \frac{\partial \xi^l}{\partial \varkappa^k} \qquad (2.4)$$

Equations (2.1),(2.2),(2.3) and (2.4) will have to be supplemented with equations for η_l^k and formulas relating η^l with $\varkappa^s$ and time. The required expressions are obtained from Equations [2]

$$\boldsymbol{\xi}_l = \int_0^t (\boldsymbol{\xi}_l \times \mathbf{u})\, dt + \boldsymbol{\xi}_l(0), \qquad \mathbf{u} = u_\eta^s \boldsymbol{\eta}_s, \qquad \boldsymbol{\xi}_l = \alpha_l^k \boldsymbol{\eta}_k \qquad (2.5)$$

Here u_η^s are projections of vector $\mathbf{u}$ of angular velocity of rotation of the Earth on η^s-axes, and α_l^k the previously introduced direction cosines between axes $\xi^1\xi^2\xi^3$ and $\eta^1\eta^2\eta^3$.

From the second equation of (2.5) and from (1.10) we find

$$\eta_l^k = \boldsymbol{\eta}_l \cdot \mathbf{r}^k = \alpha_{ls} a^{km} \frac{\partial \xi^s}{\partial \varkappa^m}, \qquad \eta^k = \alpha_s^k \xi^s \qquad (2.6)$$

Equations (2.6), in which ξ^s are given by (1.1), fully define the value of η_l^k and η^k, of the integrands of the right-hand sides of Formulas (2.1).

The contravariant components η_l^k of loci $\boldsymbol{\eta}_l$ in the basic system of coordinates and the coordinates η^k can be computed also in another way. Equations (2.5) may be written in the differential form

$$d\boldsymbol{\eta}_l / dt + \boldsymbol{\eta}_l \times \mathbf{u} = 0 \qquad (2.7)$$

From (2.7), by covariant differentiation of $\boldsymbol{\eta}_l$, followed by solution of the obtained equation with respect to $\eta_l^{k\cdot}$ and consequent integration, we find

$$\eta_l^k = -\int_0^t [\eta_l^m (\Gamma_{sm}^k \varkappa^{s\cdot} + \Gamma_{0m}^k) + (\boldsymbol{\eta}_l \times \mathbf{u})\cdot \mathbf{r}^k]\, dt + \eta_l^k(0) \qquad (2.8)$$

We shall introduce Levi-Civita symbols for the purpose of solving for the mixed products of vectors $\boldsymbol{\eta}_l$, $\mathbf{u}$ and $\mathbf{r}^k$ in the integrands of (2.8)

$$\varepsilon_{kns} = \mathbf{r}_k \cdot (\mathbf{r}_n \times \mathbf{r}_s), \qquad \varepsilon^{kns} = \mathbf{r}^k \cdot (\mathbf{r}^n \times \mathbf{r}^s) \qquad (2.9)$$

With these the mixed product $(\boldsymbol{\eta}_l \times \mathbf{u})\cdot\mathbf{r}^k$ can be written as follows:

$$(\boldsymbol{\eta}_l \times \mathbf{u})\cdot \mathbf{r}^k = \varepsilon^{ptk} \eta_{lp} u_t \qquad (2.10)$$

But $u_t = \mathbf{u}\cdot\mathbf{r}_t = u_\eta^i \eta_i^n a_{nt}$, $\eta_{lp} = \eta_l^q a_{qp}$. Therefore Equation (2.10) becomes

$$(\boldsymbol{\eta}_l \times \mathbf{u})\cdot \mathbf{r}^k = \varepsilon^{ptk} \eta_l^q \eta_i^n a_{qp} a_{nt} u_\eta^i \qquad (2.11)$$

Introducing (2.11) into (2.8) we obtain

$$\eta_l^k = -\int_0^t [\eta_l^k (\Gamma_{sm}^k \varkappa^{s\cdot} + \Gamma_{0m}^k) + \varepsilon^{ptk} \eta_l^q a_{qp} a_{nt} \eta_i^n u_\eta^i]\, dt + \eta_l^k(0) \qquad (2.12)$$

Finally, from the definitions of η^k and η_l^k, and from the evident relation $\mathbf{r} = \eta^k \boldsymbol{\eta}_k$, we have

$$\eta^k = \frac{1}{2} \eta_k^s \frac{\partial (\mathbf{r})^2}{\partial \varkappa^s} \qquad (2.13)$$

Superscript s indicates here summation from 1 to 3.

We may note that by using Levi-Civita symbols, the first group of Equations (2.5) can be solved for α_l^k

$$\alpha_l^k = \int_0^t \frac{1}{J} \varepsilon_{ijk} \alpha_l^i u_\eta^j dt + \alpha_l^k (0) \qquad (2.14)$$

where J is the Jacobian (1.1).

Thus, for the case when directions $\mathbf{e}_s$ are arbitrary functions of $\varkappa^s$ and time, the unperturbed functioning of the inertial system is defined by Equations (2.1) to (2.3), (2.5), (2.14), (2.6) and (1.11), or by equivalent equations (2.1) to (2.3), (2.12), (2.13) and (1.11). These equations determine coordinates $\varkappa^s$.

In the inertial guidance system under consideration the gyro-stabilized platform is taken as the basic kinematic scheme. The angular displacements of the gimbal rings attaching the platform to the body determine the orientation of the latter with respect to axes $\xi^1 \xi^2 \xi^3$, coincident with axes of the platform. Equations (1.2) and (1.1) define the orientation of vectors of the fundamental base, while Equations $\mathbf{e}_s = \mathbf{e}_s (\varkappa^1, \varkappa^2, \varkappa^3, t)$ define those of the sensitivity axes of the accelerometers. Therefore, the angular displacement of the platform suspension gimbal rings together with (1.2) and (1.1) determine the orientation of the body with respect to the base vectors, i.e. with respect to surfaces of $\varkappa^s = \text{const}$ coordinates, and together with $\mathbf{e}_s = \mathbf{e}_s(\varkappa^1, \varkappa^2, \varkappa^3, t)$ its orientation with respect to the axes of accelerometers.

Directions $\mathbf{e}_s$ of sensitivity axes of the model considered in Section 2 were not specified, except that these were not coplanar, and that their direction cosines with respect to ξ^k-axes were known as functions of $\varkappa^s$ and time. The integration of Equations (1.13) was carried out by means of separation of total derivatives from the sums $\varkappa^{k\cdot\cdot} e_{sk}$. Separation of variables was made after the first integration. We shall now investigate another method.

3. We shall select the directions of the sensitivity axes $\mathbf{e}_s$ so that each of the three accelerometers would register second derivatives with respect to time for one of the coordinates only. This can be achieved by selecting e_{sk} so that $e_{sk} = 0$ for $s \neq k$, and $e_{sk} = e_{ss} \neq 0$ for $s = k$. This selection means that $\mathbf{e}_s$ is normal to $\mathbf{r}_k$ when $k \neq s$, and coincides therefore with vector $\mathbf{r}^s$ of the reciprocal base. This result could have been easily predicted, as vectors $\mathbf{r}^s$ are by definition normal to surfaces of coordinates of equal magnitude, i.e. they are gradient vectors. From this follows

$$\mathbf{e}_s = \frac{\mathbf{r}^s}{\sqrt{a^{ss}}}, \qquad e_{ss} = \frac{1}{\sqrt{a^{ss}}} \qquad (3.1)$$

Taking into account (3.1), expressions (1.13) for the indications n_{e_s} of accelerometers become (do not sum with respect to s !)

$$n_{e_s} = \frac{1}{\sqrt{a^{ss}}} (\varkappa^{s\cdot\cdot} + \Gamma_{mn}^s \varkappa^{m\cdot} \varkappa^{n\cdot} + 2\Gamma_{0n}^s \varkappa^{n\cdot} + \Gamma_{00}^s - \text{grad}^l \, U \eta_l^s) \qquad (3.2)$$

In order to avoid numerical computations on accelerometer readings prior to integration of these, we shall transform Equations (3.2) by subtracting $[{}^1/_2 \varkappa^{s\cdot} (a^{ss})^\cdot] (a^{ss})^{-3/2}$. from the left and right-hand sides of these. We then have

$$\frac{d}{dt}\frac{\varkappa^{s\cdot}}{\sqrt{a^{ss}}} = n_{e_s} - \frac{1}{\sqrt{a^{ss}}}[\varkappa^{s\cdot}(\ln\sqrt{a^{ss}})^{\cdot} + \Gamma_{mn}^{s}\varkappa^{m\cdot}\varkappa^{n\cdot} + 2\Gamma_{0n}^{s}\varkappa^{n\cdot} + \Gamma_{00}^{s} - \mathrm{grad}^{l}\,U\eta_l^s] \tag{3.3}$$

Integration of (3.3) gives

$$\frac{\varkappa^{s\cdot}}{\sqrt{a^{ss}}} = \int_0^t \Big\{n_{e_s} - \frac{1}{\sqrt{a^{ss}}}[\varkappa^{s\cdot}(\ln\sqrt{a^{ss}})^{\cdot} + \Gamma_{mn}^{s}\varkappa^{m\cdot}\varkappa^{n\cdot} + 2\Gamma_{0n}^{s}\varkappa^{n\cdot} + \Gamma_{00}^{s} - \mathrm{grad}^{l}\,U\eta_l^s]\Big\}dt + \frac{\varkappa^{s\cdot}(0)}{\sqrt{a^{ss}(0)}} \tag{3.4}$$

$$\varkappa^{s\cdot} = \left(\frac{\varkappa^{s\cdot}}{\sqrt{a^{ss}}}\right)\sqrt{a^{ss}}, \qquad \varkappa^{s} = \int_0^t \varkappa^{s\cdot}\,dt + \varkappa^{s}(0)$$

Equations (2.14), (2.5) and (2.6), or (2.12) and (2.13) remain, of course, valid for the computation of η_l^s and η^s.

	ξ^1	ξ^2	ξ^3
$\mathbf{e}_s$	$\frac{\mathbf{r}^s\cdot\xi_1}{\sqrt{a^{ss}}}$	$\frac{\mathbf{r}^s\cdot\xi_2}{\sqrt{a^{ss}}}$	$\frac{\mathbf{r}^s\cdot\xi_3}{\sqrt{a^{ss}}}$

(3.5)

The orientation of directions of the sensitivity axes e_s is given by Equations (3.1), or this can be taken from a corresponding table of direction cosines.

Orientation of the object itself in relation to directions $\mathbf{e}_s$ is found from the Table (3.5) in conjunction with the angular displacement of gimbal rings of the platform.

4. We shall now consider Equations (3.4), (2.14), (2.5), (2.6) and (1.11), or (3.4), (2.12), (2.13) and (1.11) for the particular case of orthogonal coordinates $\varkappa^1, \varkappa^2, \varkappa^3$.

In this case vectors of the basic system are normal to each other. Vectors of the reciprocal basis coincide directionally with those of the basic system. Only the diagonal components a^{ss}, a_{ss} of the metric tensor are not equal to zero. These are expressed in terms of Lamé operators h_s by

$$a_{ss} = 1\,/\,a^{ss} = h_s^2 \tag{4.1}$$

For orthogonal coordinates only the following Christoffel symbols of the first and second kind [4] are different from zero:

$$\Gamma_{ss,k} = -h_s\frac{\partial h_s}{\partial \varkappa^k}, \qquad \Gamma_{sk}^{s} = \Gamma_{ks}^{s} = \frac{\partial \ln h_s}{\partial \varkappa^k}, \quad \Gamma_{sk,s} = \Gamma_{ks,s} = h_s\frac{\partial h_s}{\partial \varkappa^k}$$

$$\Gamma_{ss}^{k} = -\frac{h_s}{h_k^2}\frac{\partial h_s}{\partial \varkappa_k} \quad \Gamma_{ss,s} = h_s\frac{\partial h_s}{\partial \varkappa^s}, \quad \Gamma_{ss}^{s} = \frac{\partial \ln h_s}{\partial \varkappa^s} \tag{4.2}$$

Taking into account (4.1) and (4.2) we have

$$(\ln\sqrt{a^{ss}})^{\cdot} = -\Gamma_{sk}^{s}\varkappa^{k\cdot} - \Gamma_{0s}^{s} \tag{4.3}$$

Now Equations (3.3) can be written (do not sum with respect to s !)

$$\frac{d}{dt}(h_s\varkappa^{s\cdot}) = n_{e_s} - h_s[\Gamma_{0s}^{s}\varkappa^{s\cdot} + \Gamma_{ks}^{s}\varkappa^{k\cdot}\varkappa^{s\cdot} + \\ + \Gamma_{kk}^{s}(\varkappa^{k\cdot})^2 + 2\Gamma_{0k}^{s}\varkappa^{k\cdot} + \Gamma_{00}^{s} - \mathrm{grad}^l\, U\eta_l^s] \tag{4.4}$$

where summation is to be carried out with respect to k which is different from s. Accordong to (1.7) and (4.1)

$$\Gamma_{00,s} = \frac{\partial^2 \mathbf{r}}{\partial t^2}\cdot\mathbf{r}_s, \qquad \Gamma_{00}^{s} = \frac{1}{h_s^2}\frac{\partial^2\mathbf{r}}{\partial t^2}\cdot\mathbf{r}_s$$

$$\Gamma_{0k,s} = \frac{\partial^2\mathbf{r}}{\partial t\partial\varkappa^k}\cdot\mathbf{r}_s, \qquad \Gamma_{0k}^{s} = \frac{1}{h_s^2}\frac{\partial^2\mathbf{r}}{\partial t\partial\varkappa^k}\cdot\mathbf{r}_s \tag{4.5}$$

and for orthogonal coordinates

$$\Gamma_{0k,s} = -\Gamma_{0s,k} \tag{4.6}$$

This is because $\partial(\mathbf{r}_s\cdot\mathbf{r}_k)/\partial t = 0$. From (4.4) we find

$$h_s\varkappa^{s\cdot} = \int_0^t [n_{e_s} - h_s(\Gamma_{0s}^{s}\varkappa^{s\cdot} + \Gamma_{ks}^{s}\varkappa^{k\cdot}\varkappa^{s\cdot} + \Gamma_{kk}^{s}(\varkappa^{k\cdot})^2 + 2\Gamma_{0k}^{s}\varkappa^{k\cdot} + \Gamma_{00}^{s} - \\ - \mathrm{grad}^l\, U\eta_l^s)]\,dt + h_s(0)\,\varkappa^{s\cdot}(0)$$

$$\varkappa^{s\cdot} = \frac{1}{h_s}(\varkappa^{s\cdot}h_s), \qquad \varkappa^s = \int_0^t \varkappa^{s\cdot}\,dt + \varkappa^s(0) \tag{4.7}$$

Formulas (4.7) replace for orthogonal coordinates Formulas (3.4).

We shall now revert to Formulas (2.12), (2.13) and (2.14), (2.5), (2.6). Taking into account (4.1) and (4.2), Equations (2.12) can be simplified and presented as

$$\eta_l^k = -\int_0^t [\eta_l^k(\Gamma_{kk}^{k}\varkappa^{k\cdot} + \Gamma_{0k}^{k} + \Gamma_{mk}^{k}\varkappa^{m\cdot}) + \eta_l^m(\Gamma_{0m}^{k} + \Gamma_{km}^{k}\varkappa^{k\cdot} + \Gamma_{mm}^{k}\varkappa^{m\cdot}) + \\ + \varepsilon^{ptk}\eta_l^p\eta_i^t h_i^2 h_p^2 u_\eta^i]\,dt + \eta_l^k(0) \tag{4.8}$$

Equations (2.13), as well as (2.5) and the last of Equations (2.6) remain unchanged. The latter equation determines η^k. The first of Equations (2.6) takes the form

$$\eta_l^k = \alpha_{ls}\frac{1}{h_k^2}\frac{\partial\xi^s}{\partial\varkappa^k} \tag{4.9}$$

In the case of an orthogonal coordinate grid $J = h_1h_2h_3$ Formulas (2.14) will have the form

$$\alpha_l^k = \int_0^t \frac{1}{h_ih_jh_k}\varepsilon_{ijk}\alpha_l^i u_\eta^j\,dt + \alpha_l^k(0) \tag{4.10}$$

and the direction cosines (3.5) are given in the table (4.11).

(4.11)

	ξ^1	ξ^2	ξ^3
$\mathbf{e}_s$	$\frac{1}{h_s}\frac{\partial\xi^1}{\partial\varkappa^s}$	$\frac{1}{h_s}\frac{\partial\xi^2}{\partial\varkappa^s}$	$\frac{1}{h_s}\frac{\partial\xi^3}{\partial\varkappa^s}$

In the case of an orthogonal curvilinear system of coordinates, it is evidently possible to design an inertial guidance system based on a controllable gyro-frame, as in this case directions $\mathbf{e}_s$ form a rigid orthogonal trihedron which can be the trihedron of the gyro-frame platform.

In order to keep the trihedron of the gyro-frame platform coincident with the trihedron formed by vectors $\mathbf{r}_s$ of the fundamental base, moments will have to be applied to the gyroscopes of this frame. It is assumed that at the initial moment the two trihedrons coincide.

For the formulation of these control moments we shall need expressions for the projections $\omega_{(s)}$ of the vector of absolute angular velocity of the trihedron $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$ on its edges expressed as functions of coordinates $\varkappa^s$, their derivatives, and time. These formulas are derived below.

5. We shall introduce notations $\mathbf{p}_s = d\mathbf{r}_s/dt$, and shall represent vectors $\mathbf{p}_s$ as follows:

$$\mathbf{p}_s = \frac{\partial^2 \mathbf{r}}{\partial \varkappa^s \partial \varkappa^n} \varkappa^{n\cdot} + \frac{\partial^2 \mathbf{r}}{\partial t\, \partial \varkappa^s} \tag{5.1}$$

From the definitions of Christoffel and of the $\Gamma_{0s}^{\bullet}$ symbols

$$\frac{\partial^2 \mathbf{r}}{\partial \varkappa^s \partial \varkappa^n} = \Gamma_{sn}^m \mathbf{r}_m, \qquad \frac{\partial^2 \mathbf{r}}{\partial t \partial \varkappa^s} = \Gamma_{0s}^m \mathbf{r}_m \tag{5.2}$$

Therefore

$$\mathbf{p}_s = (\Gamma_{sn}^m \varkappa^{n\cdot} + \Gamma_{0s}^m)\, \mathbf{r}_m \tag{5.3}$$

Let p_{sk} and p_s^k denote respectively the covariant and contravariant components of $\mathbf{p}_s$ with respect to vectors $\mathbf{r}_k$ of the fundamental base. From (5.3) we find

$$p_s^k = \Gamma_{sn}^k \varkappa^{n\cdot} + \Gamma_{0s}^k, \qquad p_{sk} = a_{mk} p_s^m \tag{5.4}$$

On the other hand, (do not sum with respect to s !)

$$\frac{d\mathbf{r}_s}{dt} = \frac{d}{dt}\left[\sqrt{a_{ss}}\left(\frac{\mathbf{r}_s}{\sqrt{a_{ss}}}\right)\right] \tag{5.5}$$

where $\mathbf{r}_s / \sqrt{a_{ss}}$ are unit vectors of ditections $\mathbf{r}_s$. Consequently

$$\mathbf{p}_s = \frac{(\sqrt{a_{ss}})^{\cdot}}{\sqrt{a_{ss}}} \mathbf{r}_s + \sqrt{a_{ss}} \frac{d}{dt} \frac{\mathbf{r}_s}{\sqrt{a_{ss}}} \tag{5.6}$$

As vectors $\mathbf{r}_s$ of the fundamental base form a rigid trihedron, we have

$$\frac{d}{dt} \frac{\mathbf{r}_s}{\sqrt{a_{ss}}} = \frac{\boldsymbol{\omega} \times \mathbf{r}_s}{\sqrt{a_{ss}}} \tag{5.7}$$

where $\boldsymbol{\omega}$ is the absolute angular velocity of this trihedron.

Substituting (5.7) into (5.6), we obtain expressions for vectors $\mathbf{p}_s$ in terms of the absolute angular velocity of rotation of the base

$$\mathbf{p}_s = \frac{(\sqrt{a_{ss}})^{\cdot}}{\sqrt{a_{ss}}} \mathbf{r}_s + \boldsymbol{\omega} \times \mathbf{r}_s, \qquad p_{sk} = \frac{a_{sk}(\sqrt{a_{ss}})^{\cdot}}{\sqrt{a_{ss}}} + (\boldsymbol{\omega} \times \mathbf{r}_s)\cdot \mathbf{r}_k \tag{5.8}$$

With orthogonal $\mathbf{r}_s$ the nondiagonal components of the metric tensor are zero, therefore for $s \neq k$ Equations (5.8) become

$$p_{sk} = (\boldsymbol{\omega} \times \mathbf{r}_s)\cdot \mathbf{r}_k \quad \text{or} \quad p_{sk} = \omega^n (\mathbf{r}_n \times \mathbf{r}_s)\cdot \mathbf{r}_k \tag{5.9}$$

Using Levi-Civita symbols we arrive at Equations $p_{sk} = \omega^n \varepsilon_{nsk}$, from which, after multiplication by ε^{nsk} .(do not sum with respect to s, k !), we find the contravariant components ω^n of vector $\boldsymbol{\omega}$ in the fundamental base

$$\omega^n = \varepsilon^{nsk} p_{sk} \tag{5.10}$$

Reverting to (5.4), we find expressions ω^n and ω_l in terms of $\varkappa^s$, derivatives of these and time

$$\omega^n = \varepsilon^{nsk}(\Gamma_{sm,k}\varkappa^{m\cdot} + \Gamma_{0s,k}), \qquad \omega_l = a_{ln}\omega^n \tag{5.11}$$

Required projections $\omega_{(s)}$ on directions $\mathbf{r}_s$ are easily found from known covariant components ω_s of vector $\boldsymbol{\omega}$

$$\omega_{(s)} = \frac{\omega_s}{\sqrt{a_{ss}}} = \sqrt{a_{ss}}\varepsilon^{snk}(\Gamma_{nm,k}\varkappa^{m\cdot} + \Gamma_{0n,k}) \tag{5.12}$$

In (5.12) indexes n and k are different. From (4.2) it follows that Christoffel symbols in (5.12) are not zero, only when either $n = m$, or $k = m$. As according to (4.1) $h_s = \sqrt{a_{ss}}$, and $J = h_1 h_2 h_3$, and with Levi-Civita symbols $\varepsilon^{snk} = \pm 1 / J$, where the sign of the right-hand side is determined by the order of indexes s, n, k, we obtain from (5.12)

$$\omega_{(1)} = \frac{1}{h_2 h_3}(\Gamma_{2m,3}\varkappa^{m\cdot} + \Gamma_{02,3}) = -\frac{1}{h_2 h_3}(\Gamma_{3m,2}\varkappa^{m\cdot} + \Gamma_{03,2}) \qquad (123) \tag{5.13}$$

Here the first of the right-hand side expressions correspond to the order of indexes s, n, k : 1 2 3, 2 3 1, 3 1 2, and the second to 1 3 2, 2 1 3, 3 2 1. The two different expressions of Formulas (5.13) for each projection $\omega_{(s)}$ are identical, as it follows from (4.2) and (4.6) that $\Gamma_{sk,s} = -\Gamma_{ss,k}$ and $\Gamma_{0k,s} = -\Gamma_{0s,k}$.

In the particular case of $\Gamma_{0s,k}$ equal zero, i.e. when the set of coordinates $\varkappa^s$ does not change its position with time in the coordinate system $O_1\xi^1\xi^2\xi^3$, we have

$$\omega_{(1)} = \frac{1}{h_2 h_3}\Gamma_{2m,3}\varkappa^{m\cdot} = -\frac{1}{h_2 h_3}\Gamma_{3m,2}\varkappa^{m\cdot} \qquad (123) \tag{5.14}$$

In this case, using expressions given in (4.2) for representation of Christoffel symbols by Lamé operators, we arrive at the following expressions for $\omega_{(s)}$:

$$\omega_{(1)} = -\frac{1}{h_3}\frac{\partial h_2}{\partial \varkappa^3}\varkappa^{2\cdot} + \frac{1}{h_2}\frac{\partial h_3}{\partial \varkappa^2}\varkappa^{3\cdot} \qquad (123)$$

6. So far, in dealing with nonstationary curvilinear coordinates, we have been considering the general case in which the coordinate surfaces $\varkappa^s$ = const could arbitrarily change their position with time, relative to the trihedron $O_1\xi^1\xi^2\xi^3$. This is seen from Formulas (1.1) which contain the parameter of time explicitly. However, the character of this dependence was not specified.

One particular case of such dependence is of special interest. It is the case in which the curvilinear coordinates $\varkappa^s$, defining the position of the object in the coordinate system $O_1\eta^1\eta^2\eta^3$ rigidly tied to the Earth, are stationary with respect to the trihedron $O_1\eta^1\eta^2\eta^3$. Then

$$\eta^s = \eta^s(\varkappa^1, \varkappa^2, \varkappa^3) \tag{6.1}$$

The $\varkappa^s$ coordinates are not stationary in relation to the basic system of Cartesian coordinates $O_1\xi^1\xi^2\xi^3$.

We shall assume that the angular velocity of rotation of the Earth is constant and that its direction does not change with respect to the system of coordinates $O_1\xi^1\xi^2\xi^3$. We note that so far no use has been made of this assumption.

Let axis $O_1\xi^3$ of the coordinate system $O_1\xi^1\xi^2\xi^3$ coincide with $O_1\eta^3$, and the latter, in turn, coincide with vector $\mathbf{u}$ of the angular velocity of the rotation of the Earth. We shall also assume that at the initial moment ($t = 0$) the two trihedrons $O_1\xi^1\xi^2\xi^3$ and $O_1\eta^1\eta^2\eta^3$ fully coincide. Then, for any instant

$$\xi^1 = \eta^1\cos ut - \eta^2\sin ut, \qquad \xi^2 = \eta^1\sin ut + \eta^2\cos ut, \qquad \xi^3 = \eta^3 \tag{6.2}$$

For the computation of components of the metric tensor, Lamé operators and Christoffel symbols, we can assume in this case t to be equal to zero. The validity of this statement follows from the fact that the set of $\varkappa^s$ coordinates moves as a single entity (rotates with angular velocity **u**), and that the properties of the space defined by these coordinates are independent of time. This, of course, can be established also by direct calculation.

From (1.7) and (6.2) we derive for $\Gamma_{0k,s}$ and $\Gamma_{00,s}$ expressions which are also independent of time

$$\Gamma_{0k,\,s} = u\left(\frac{\partial\eta^1}{\partial\varkappa^k}\frac{\partial\eta^2}{\partial\varkappa^s} - \frac{\partial\eta^2}{\partial\eta^k}\frac{\partial\eta^1}{\partial\varkappa^s}\right), \qquad \Gamma_{00,\,s} = -\frac{u^2}{2}\frac{\partial}{\partial\varkappa^s}\left[(\eta^1)^2 + (\eta^2)^2\right] \tag{6.3}$$

7. We shall consider in conclusion two examples, the case of Cartesian coordinates $\varkappa^s = \xi^s$, and that of geocentric coordinates $\varkappa^1 = r$, $\varkappa^2 = \lambda$, $\varkappa^3 = \varphi$, when

$$\eta^1 = r\cos\varphi\cos\lambda, \qquad \eta^2 = r\cos\varphi\sin\lambda, \qquad \eta^3 = r\sin\varphi \tag{7.1}$$

We shall proceed from Formulas (4.8), (4.7), (2.13) and Table (4.11).

In the case of Cartesian coordinates $\varkappa^s = \xi^s$, all of the Christoffel symbols and Γ_{0k}^s, Γ_{00}^s, $\Gamma_{0k,s}$, $\Gamma_{00,s}$ are zero. The Lamé operators h_s are equal to unity. From (4.7) and (4.8) we have

$$\xi^{s\cdot} = \int_0^t (n_{e_s} + \mathrm{grad}^l U\eta_l^s)\,dt + \xi^{s\cdot}(0), \quad \xi^s = \int_0^t \xi^{s\cdot}dt + \xi^s(0) \tag{7.2}$$

$$\eta_l^k = -\int_0^t \varepsilon^{ptk}\eta_l^p\eta_i^t u_\eta^i dt + \eta_l^k(0) \tag{7.3}$$

Assuming as in Section 6 the coincidence of the two trihedrons $\xi^1\xi^2\xi^3$ and $\eta^1\eta^2\eta^3$ at the initial moment, and noting that $u_\eta^3 = u$, $n_\eta^1 = u_\eta^2 = 0$, when axis $O_1\eta^3$ coincides with the axis of rotation of the Earth, we obtain from (7.3) $\eta_3^1 = \eta_3^2 = \eta_1^3 = \eta_2^3 = 0$, $\eta_3^3 = 1$. For the remaining η_l^k we have two sets of differential equations of the second order

$$\eta_1^1 = -\int_0^t \eta_1^2 u dt + 1, \qquad \eta_1^2 = \int_0^t \eta_1^1 u dt; \qquad \eta_1^2 = -\int_0^t \eta_2^2 u dt, \qquad \eta_2^2 = \int_0^t \eta_2^1 u dt + 1 \tag{7.4}$$

Assuming as in Section 6 $u = \mathrm{const}$, we find from (7.4)

$$\eta_1^1 = \cos ut, \qquad \eta_2^1 = -\sin ut, \qquad \eta_1^2 = \sin ut, \qquad \eta_2^2 = \cos ut \tag{7.5}$$

From (2.13) we have $\eta^k = \eta_k^s\xi^s$, and from (7.5)

$$\eta^1 = \xi^1\cos ut + \xi^2\sin ut, \qquad \eta^2 = -\xi^1\sin ut + \xi^2\cos ut, \qquad \eta^3 = \xi^3$$

which is in agreement with (6.2). Using (4.10), (4.9) and (2.6) for (4.8) and (2.13) we find $\alpha_l^k = \eta_l^k$, and from (4.10) we obtain for α_l^k, the same expressions as for η_l^k. After this, from (2.6) we arrive again at Equations (7.5).

We shall now turn to the second example. Here the Lamé operators are

$$h_1 = 1, \qquad h_2 = r\cos\varphi, \qquad h_3 = r \tag{7.6}$$

Christoffel symbols of the second kind which differ from zero, are

$$\Gamma_{21}^2 = \Gamma_{12}^2 = \frac{1}{r}, \qquad \Gamma_{23}^2 = \Gamma_{32}^2 = -\tan\varphi, \qquad \Gamma_{31}^3 = \Gamma_{13}^3 = \frac{1}{r}, \qquad \Gamma_{22}^1 = -r\cos^2\varphi$$
$$\Gamma_{22}^3 = \sin\varphi\cos\varphi, \qquad \Gamma_{33}^1 = -r \tag{7.7}$$

In accordance with (6.3) and (7.1), the following symbols are also nonzero

$$\Gamma_{00}^1 = -u^2 r\cos^2\varphi, \qquad \Gamma_{00}^3 = u^2\sin\varphi\cos\varphi, \qquad \Gamma_{01}^2 = u/r$$
$$\Gamma_{02}^1 = -ur\cos^2\varphi, \qquad \Gamma_{02}^3 = u\sin\varphi\cos\varphi, \qquad \Gamma_{03}^2 = -u\tan\varphi \tag{7.8}$$

Taking into account (7.6), (7.7) and (7.8), Equations (4.7) can be presented in the form

$$r^{\cdot} = \int_0^t [n_1 + r(\varphi^{\cdot 2} + (u+\lambda^{\cdot})^2 \cos^2\varphi) + \operatorname{grad}^l U \eta_l^1]\, dt + r^{\cdot}(0) \tag{7.9}$$

$$r\cos\varphi\lambda^{\cdot} = \int_0^t \left[n_2 - (\lambda^{\cdot} + 2u)(r^{\cdot}\cos\varphi - r\varphi^{\cdot}\sin\varphi) + \frac{1}{r\cos\varphi}\operatorname{grad}^l U\eta_l^2\right] dt + r(0)\cos\varphi(0)\,\lambda^{\cdot}(0)$$

$$r\varphi^{\cdot} = \int_0^t \left[n_3 - (r^{\cdot}\varphi^{\cdot} + r\sin\varphi\cos\varphi(\lambda^{\cdot} + u)^2) + \frac{1}{r}\operatorname{grad}^l U\eta_l^3\right] dt + r(0)\,\varphi^{\cdot}(0)$$

$$= \int_0^t r^{\cdot} dt + r(0), \qquad \lambda = \int_0^t \frac{1}{r\cos\varphi}(r\cos\varphi\lambda^{\cdot})\, dt + \lambda(0), \qquad \varphi = \int_0^t \frac{1}{r}(r\varphi^{\cdot})\, dt + \varphi(0)$$

And again taking into account (7.6), (7.7) and (7.8), Equations (4.8) can be presented in the form

$$\eta_l^1 = -\int_0^t [-\eta_l^2 r\cos^2\varphi(\lambda^{\cdot} + u) - \eta_l^3 r\varphi^{\cdot} + u(\eta_l^2\eta_3^3 - \eta_l^3\eta_3^2)\, r^2\cos\varphi]\, dt + \eta_l^1(0)$$

$$\eta_l^2 = -\int_0^t \Big[\eta_l^1 \frac{(\lambda^{\cdot} + u)}{r} - \eta_l^3(\lambda^{\cdot} + u)\tan\varphi + \eta_l^2\left(\frac{r^{\cdot}}{r} - \tan\varphi\varphi^{\cdot}\right) + u(\eta_l^3\eta_3^1 - \eta_l^1\eta_3^3)\frac{1}{\cos\varphi}\Big] dt + \eta_l^2(0) \tag{7.10}$$

$$\eta_l^3 = -\int_0^t \left[\eta_l^2 \sin\varphi\cos\varphi(\lambda^{\cdot} + u) + \eta_l^1\frac{\varphi^{\cdot}}{r} + \eta_l^3\frac{r^{\cdot}}{r} + u\cos\varphi(\eta_l^1\eta_3^2 - \eta_l^2\eta_3^1)\right] dt + \eta_l^3(0)$$

For u = const the following values of η_l^s satisfy Equations (7.10)

$$\begin{aligned}
&\eta_1^1 = \cos\varphi\cos\lambda, && \eta_1^2 = -\frac{\sin\lambda}{r\cos\varphi}, && \eta_1^3 = -\frac{\sin\varphi\cos\lambda}{r}\\
&\eta_2^1 = \cos\varphi\sin\lambda, && \eta_2^2 = \frac{\cos\lambda}{r\cos\varphi}, && \eta_2^3 = -\frac{\sin\varphi\sin\lambda}{r}\\
&\eta_3^1 = \sin\varphi, && \eta_3^2 = 0, && \eta_3^3 = \frac{\cos\varphi}{r}
\end{aligned} \tag{7.11}$$

It may be noted that for u = const, the value of η_s^k can be computed directly from the definition $\eta_l^s = \eta_l \cdot \mathbf{r}_s / h_2$ and from Equations (7.1) and $\eta^k = \eta_k^s\xi^s$, without recourse to (4.8). This may be used as a proof of correctness of calculations.

Expressions for $\eta^1\eta^2\eta^3$ coincident with those of (7.1) are obtained from (2.13) and (7.10).

Equations for computing α_l^k are obtained from (4.10), (7.5) and (7.3) by using Formulas (4.9) and (2.6) for (4.8) and (2.13)

$$\alpha_l^1 = \int_0^t \alpha_l^2\, u dt + \alpha_l^1(0), \quad \alpha_l^2 = -\int_0^t \alpha_l^1\, u dt + \alpha_l^2(0), \quad \alpha_l^3 = \alpha_l^3(0) \tag{7.12}$$

From these we obtain for α_l^k, the same values, as in the case of Cartesian coordinates $x^s = \xi^s$, for η_l^k. From Equations $\eta^k = \eta_k^s\xi^s$ we find $\xi^1 = r\cos\varphi\cos(\lambda + ut)$, $\xi^2 = r\cos\varphi\sin(\lambda + ut)$, $\xi^3 = r\sin\varphi$. After that, with reference to (4.11) and (7.6), we obtain the direction cosines of sensitivity axes of accelerometers as follows:

	ξ^1	ξ^2	ξ^3
$\mathbf{e}_1$	$\cos\varphi\cos(\lambda+ut)$	$\cos\varphi\sin(\lambda+ut)$	$\sin\varphi$
$\mathbf{e}_2$	$-\sin(\lambda+ut)$	$\cos(\lambda+ut)$	0
$\mathbf{e}_3$	$-\sin\varphi\cos(\lambda+ut)$	$-\sin\varphi\sin(\lambda+ut)$	$\cos\varphi$

Finally, from (5.13), (7.6), (7.7) and (7.8), we find

$$\omega_{(1)} = (u + \lambda^{\cdot})\sin\varphi, \qquad \omega_{(2)} = -\varphi^{\cdot}, \qquad \omega_{(3)} = (u + \lambda^{\cdot})\cos\varphi$$

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